Math 1552

Section 10.6: Alternating Series

Math 1552 lecture slides adapted from the course materials
By Klara Grodzinsky (GA Tech, *School of Mathematics*, Summer 2021)

Alternating Series Test

Let $\sum_{k} a_k$ be an alternating series.

$b_k \geq 0$ for all k. Then an alternating series looks like $\sum_k (-1)^k b_k$ $(a_k = (-1)^k b_k)$

(a) If $\sum_{k} |a_k|$ converges, then the series *converges absolutely*.

Alternating Series Test (cont.)

$\sum_k (-1)^k b_k$, where $b_k \geq 0$

Let $\sum_{k} a_k$ be an alternating series.

, e.g., we don't have absolute conv. of the series

(b) If (a) fails, then if :

i) $\{|a_n|\}$ is a decreasing sequence, and

ii) $\lim_{n\to\infty} |a_n| = 0$,

then the series *converges conditionally*.

(c) Otherwise, the series *diverges*.

Example A:

Determine if the alternating series converges ~~absolutely~~, converges conditionally, or ~~diverges~~.

$$S = \sum_{k=1}^{\infty} (-1)^k \frac{1}{\sqrt{k+4}}$$

$$S = \sum_{k=1}^{\infty} (-1)^k b_k, \text{ where } b_k = \frac{1}{\sqrt{k+4}}.$$

$\rightarrow$ first, need to check for absolute convergence!

does $\sum_{k=1}^{\infty} b_k$ converge?

apply the LCT to show that we do not get absolute convergence.

→ intuition for why we do not get abs. conv:

$b_k = \dfrac{1}{k^{1/2}\sqrt{1+\frac{1}{k}}}$ ~ looks "almost" like a p-series with $p = 1/2$

→ apply the LCT, comparing to $c_k = \dfrac{1}{k^{1/2}}$.

Then $\lim\limits_{k\to\infty} \dfrac{b_k}{c_k} = \lim\limits_{k\to\infty} \dfrac{\cancel{k^{1/2}}}{\cancel{k^{1/2}}\sqrt{1+1/k}} = 1 > 0$

so both $\sum_k b_k$ and $\sum_k c_k$ diverge.

→ To see if S converges conditionally, we apply the alternating series test (AST):

- we need to check that b_k is decreasing ✓

$$\frac{1}{\sqrt{(k+1)+4}} = b_{k+1} < b_k = \frac{1}{\sqrt{k+4}} \text{ for all } k \geq 1$$

- $\lim\limits_{k\to\infty} b_k = \lim\limits_{k\to\infty} \frac{1}{\sqrt{k+4}} = 0$ ✓

- So by the AST, we see that the series converges conditionally.

Example B:

Determine if the alternating series converges absolutely, converges conditionally, or diverges.

$$\sum_{k=1}^{\infty} (-1)^k \frac{k}{3^k} = S$$

$\rightarrow S = \sum_{k=1}^{\infty} (-1)^k b_k$, for $b_k = \frac{k}{3^k} \geq 0$

for all $k \geq 1$.

$\rightarrow$ check for absolute convergence:

apply the ratio test

$$L = \lim_{N \to \infty} \frac{b_{N+1}}{b_N} = \lim_{N \to \infty} \frac{(N+1)}{3^{N+1}} \cdot \frac{3^N}{N}$$

$$= \lim_{n\to\infty} \frac{1}{3}\cdot\frac{(n+1)}{n} = \frac{1}{3} < 1$$

So since $L<1$, the series converges absolutely.

$\rightarrow$ We can stop here since we also get conditional convergence whenever we have absolute convergence.

Example C:

Determine if the alternating series converges ~~absolutely~~, converges ~~conditionally~~, or diverges. $\checkmark$

$$S=\sum_{k=1}^{\infty}(-1)^{k+1}\frac{k^3}{k^3+2k+1}$$

$\rightarrow$ $S=\sum_{k=1}^{\infty}(-1)^{k+1} b_k$, where $b_k=\dfrac{k^3}{k^3+2k+1}$

$\rightarrow$ check for absolute convergence:

apply the n^{th} term test:

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty}\frac{n^3}{n^3+2n+1} = 1 \neq 0$$

so the series $\sum_{k=1}^{\infty} b_k$ diverges, i.e., we do not

get absolute convergence

→ to check for conditional convergence, we apply the AST:

- by the nth term test (as above), we do not have that $\lim_{n \to \infty} b_n = 0$
- so the series diverges.

Estimating an Alternating Sum

Let $\sum_k a_k$ be a convergent alternating series with a sum of $\cancel{L}\ S$.

$$\sum_{k=0}^{\infty} (-1)^k b_k, \text{ for}$$

$b_k \geq 0$ at all $k \geq 0$.

nth partial sum of the alternating series

Then : $\left|s_n - \cancel{L}\ S\right| < \left|a_{n+1}\right|$.

This is the same as

$$\left| \sum_{k=0}^{N} (-1)^k b_k - \cancel{L}\ S \right| < b_{N+1}$$

$\boxed{\text{Note that in the next example, we call } L \text{ by } S \text{ instead}}$

Example:

Estimate the sum of the series below within an error range of 0.001.

$$S = \sum_{k=0}^{\infty} (-1)^k \frac{1}{(2k+1)!}$$

→ this is an alternating series:

$$\sum_{k=0}^{\infty} (-1)^k \cdot b_k, \text{ where}$$

$$b_k = \frac{1}{(2k+1)!}$$

→ we apply the approximation rule in the last slide to find a suitable n:

$$b_{N+1} = \frac{1}{(2(N+1)+1)!} = \frac{1}{(2N+3)!} < \frac{1}{1000}$$

Check: $N=0$ $\quad \frac{1}{3!} = \frac{1}{6} \geq \frac{1}{1000}$ ✗

$N=1$ $\quad \frac{1}{5!} = \frac{1}{120} \geq \frac{1}{1000}$ ✗

$N=2$ $\quad \frac{1}{7!} = \frac{1}{5040} < \frac{1}{1000}$ ✓

So we conclude that

$$\left|\sum_{k=0}^{2}(-1)^k\cdot\frac{1}{(2k+1)!}-S\right|<\frac{1}{7!}<\frac{1}{1000}$$

Hence, our approximation to S within an error of $\frac{1}{1000}$ is given by

$$\sum_{k=0}^{2}(-1)^k\cdot\frac{1}{(2k+1)!}=\frac{1}{1!}-\frac{1}{3!}+\frac{1}{5!}$$

$$=1-\frac{1}{6}+\frac{1}{120}$$

Rearrangements

- If an alternating series converges *absolutely*, rearranging the terms will NOT change the sum.

- If an alternating series converges *conditionally*, then the sum changes when the terms are written in a different order.

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Section 10.6: Alternating Series Review

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Review Question:
The series:

$$\sum_{k=0}^{\infty}(-1)^k \frac{1}{\sqrt{k^2+1}} = S$$

A. ~~Converges absolutely~~
B. Converges conditionally
C. ~~Diverges~~

$\rightarrow S = \sum_{k=0}^{\infty} (-1)^k b_k$, where $b_k = \frac{1}{\sqrt{k^2+1}} \geq 0$ for all $k \geq 0$

$\rightarrow$ check for absolute convergence:

- intuition is that $\sum_{k=0}^{\infty} b_k$ looks "almost" like

Countdown

The harmonic series (p-series with $p=1$), so we should expect it to diverge.

- use the LCT to see that it indeed diverges.

Compare to $c_k = \frac{1}{k}$ for $k \geq 1$.

$$\lim_{k\to\infty} \frac{b_k}{c_k} = \lim_{k\to\infty} \frac{k}{\sqrt{k^2+1}}$$

$$= \lim_{k\to\infty} \frac{\cancel{k}^{\,1}}{\cancel{k}\sqrt{1+1/k^2}} = 1 > 0$$

Hence by the LCT, both $\sum_k b_k$ and $\sum_k c_k$ diverge.

→ what about conditional convergence?

we apply the AST:

- check that b_k is decreasing! ✓

$$b_{k+1} = \frac{1}{\sqrt{(k+1)^2+1}} < b_k = \frac{1}{\sqrt{k^2+1}},$$

for all $k \geq 0$

- check that

$$\lim_{k\to\infty} b_k = \lim_{k\to\infty} \frac{1}{\sqrt{k^2+1}} = 0 \checkmark$$

- so by the AST, we get conditional convergence.

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Section 10.7: Power Series

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Learning Goals

- Recognize the general forms of a power series
- Understand that a power series is an "infinite polynomial"
- Determine the radius and interval of convergence for a power series
- Differentiate and integrate a power series to obtain a new power series
- later: talk about how to approximate a power series accurately to within some margin of error.

Power Series

A *power series* is an "*infinite polynomial*" and a *function* of *x*:

$$\text{Power series in } x: \; f(x) = \sum_{k=0}^{\infty} a_k x^k$$, a_k – sequence

$$\text{Power series in } x\text{-}c: \; f(x) = \sum_{k=0}^{\infty} a_k (x-c)^k,$$ depends on x and on the point c.

Series expansion in the variable x about the point c

Convergence of Power Series

Suppose that

$\sum_{k=0}^{\infty} a_k (x-c)^k$ converges at x_0, e.g., The series is convergent for $x = x_0$.

if $\sum_{k=0}^{\infty} a_k (x_0-c)^k$ converges.

The series is said to converge~~s~~ on (x_0, x_1)

if it converges at every point in the interval.

Interval of convergence

The *interval of convergence* of a power series is the set of all values of x for which the series converges.

This interval may be closed, open, or half-open. for $a < b$

$[a, b]$ (a, b)

$[a, b)$ or $(a, b]$

Question: On which interval do you think this series converges? *(Why?)*

$$\sum_{k=1}^{\infty} \frac{x^k}{k}$$